Riemann Integration

* Upper & Lower Riemann Sums:-

Let $f:[a,b]\to \mathbb{R}$ be a bounded function and $P=\{a=x_0, x_1, x_2, \ldots\ldots x_n=b\}$ be a partion of $[a,b]$. since, f is bounded on $[a,b]$, f is also bounded on each of the sub intervals.

Let M, m be the supremum and Infimum of f in $[a,b]$ and M_r, m_r be the supremum and infimum of f in the r^{th} subinterval $I_r=[x_{r-1}, x_r]$ where $r=1,2,\ldots n$.

The sum $M_1\delta_1+M_2\delta_2+\ldots\ldots+M_n\delta_n=\sum_{r=1}^{n} M_r\delta_r$ is defined as the upper Riemann sum (or) Upper Darboux's sum of f corresponding to the partition P and is denoted by $U(P,f)$.

The sum $m_1\delta_1+m_2\delta_2+\ldots\ldots+m_n\delta_n=\sum_{r=1}^{n} m_r\delta_r$ is defined as the Lower Riemann Sum (or) Lower Darboux's sum of f corresponding to the partition P and is denoted by $L(P,f)$.

Thus, we have

$$\boxed{\begin{aligned} U(P,f) &= \sum_{r=1}^{n} M_r\delta_r \text{ and} \\ L(P,f) &= \sum_{r=1}^{n} m_r\delta_r. \end{aligned}}$$

1. If $f(x)=x$ on $[0,1]$ and $P=\{0,\frac{1}{3},\frac{2}{3},1\}$ then find $U(P,f)$ and $L(P,f)$.

Sol:- Sub intervals are $I_1=[0,\frac{1}{3}]$; $I_2=[\frac{1}{3},\frac{2}{3}]$; $I_3=[\frac{2}{3},1]$

Length of each subinterval is $\delta_1=\delta_2=\delta_3=\frac{1}{3}$

Supremum and Infimum of $I_1 \Rightarrow M_1=\frac{1}{3}$; $m_1=0$

$I_2 \Rightarrow M_2=\frac{2}{3}$; $m_2=\frac{1}{3}$

$I_3 \Rightarrow M_3=1$; $m_3=\frac{2}{3}$

$$U(P,f)=\sum_{r=1}^{n} M_r\delta_r = M_1\delta_1+M_2\delta_2+M_3\delta_3 = \frac{1}{3}\left(\frac{1}{3}\right)+\frac{2}{3}\left(\frac{1}{3}\right)+1\left(\frac{1}{3}\right)$$

$$=\frac{1}{9}+\frac{2}{9}+\frac{3}{9}=\frac{6}{9}=2/3$$

$$L.(P,f)=\sum_{r=1}^{n} m_r\delta_2 = m_1\delta_1+m_2\delta_2+m_3.\delta_3 = 0\left(\frac{1}{3}\right)+\frac{1}{3}\left(\frac{1}{3}\right)+\frac{2}{3}\left(\frac{1}{3}\right)$$

$$=0+\frac{1}{9}+\frac{2}{9}=\frac{3}{9}=1/3$$

2. If $f(x)=x^2$ on $[0,1]$ and $P=\{0,\frac{1}{4},\frac{2}{4},\frac{3}{4},1\}$ then find $U(P,f)$ and $L(P,f)$.

Sol:- Sub intervals are $I_1=[0,\frac{1}{4}]$; $I_2=[\frac{1}{4},\frac{2}{4}]$; $I_3=[\frac{2}{4},\frac{3}{4}]$; $I_4=[\frac{3}{4},\frac{4}{4}]$.

Length of each subinterval is $\delta_1=\delta_2=\delta_3=\delta_4=\frac{1}{4}$

Supremum and Infimum of $I_1 \Rightarrow M_1=\frac{1}{16}$; $m_1=0$

$I_2 \Rightarrow M_2=\frac{4}{16}$; $m_2=\frac{1}{16}$

$I_3 \Rightarrow M_3=\frac{9}{16}$; $m_3=\frac{4}{16}$

$I_4 \Rightarrow M_4=1$; $m_4=\frac{9}{16}$

$$U(P,f)=\sum_{r=1}^{n} M_r\delta_r = M_1\delta_1+M_2\delta_2+M_3\delta_3+M_4\delta_4 = \frac{1}{16}\left(\frac{1}{4}\right)+\frac{4}{16}\left(\frac{1}{4}\right)+\frac{9}{16}\left(\frac{1}{4}\right)+1\left(\frac{1}{4}\right)$$

$$=\frac{1+4+9+16}{64}=\frac{30}{64}=\frac{15}{32}$$

$$L(P,f)=\sum_{r=1}^{n} m_r\delta_r = m_1\delta_1+m_2\delta_2+m_3\delta_3+m_4\delta_4 = 0\left(\frac{1}{4}\right)+\frac{1}{16}\left(\frac{1}{4}\right)+\frac{4}{16}\left(\frac{1}{4}\right)+\frac{9}{16}\left(\frac{1}{4}\right)$$

$$=\frac{0+1+4+9}{64}=\frac{14}{64}=\frac{7}{32}.$$

3\. If $f(x)=\frac{1}{x}$ on $[1,2]$ and $P=\{1,1.2,1.4,1.6,1.8,2\}$

Sol: Sub intervals are $I_1=[1,1.2]$; $I_2=[1.2,1.4]$; $I_3=[1.4,1.6]$; $I_4=[1.6,1.8]$;
$I_5=[1.8,2]$.

Length of each sub interval is $\delta_1=\delta_2=\delta_3=\delta_4=\delta_5=0.2$

supremum and Infimum of $I_1 \Rightarrow M_1=1$; $m_1=\frac{1}{1.2}$

$I_2 \Rightarrow M_2=\frac{1}{1.2}$; $m_2=\frac{1}{1.4}$

$I_3 \Rightarrow M_3=\frac{1}{1.4}$; $m_3=\frac{1}{1.6}$

$I_4 \Rightarrow M_4=\frac{1}{1.6}$; $m_4=\frac{1}{1.8}$

$I_5 \Rightarrow M_5=\frac{1}{1.8}$; $m_5=\frac{1}{2}$

$$U(P,f)=\sum_{r=1}^{n} M_r\delta_r = M_1\delta_1+M_2\delta_2+M_3\delta_3+M_4\delta_4+M_5\delta_5$$

$$=1(0.2)+\frac{1}{1.2}(0.2)+\frac{1}{1.4}(0.2)+\frac{1}{1.6}(0.2)+\frac{1}{1.8}(0.2)$$

$$=\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}=\frac{11}{30}+\frac{15}{56}+\frac{1}{9}=\frac{33+10}{90}+\frac{15}{56}$$

$$U(P,f)=\frac{43}{90}+\frac{15}{56}=\frac{2408+1350}{5040}=\frac{3758}{5040}=0.7456$$

$$L(P,f)=\sum_{r=1}^{n} m_r\delta_r = m_1\delta_1+m_2\delta_2+m_3\delta_3+m_4\delta_4+m_5\delta_5$$

$$=\frac{1}{1.2}(0.2)+\frac{1}{1.4}(0.2)+\frac{1}{1.6}(0.2)+\frac{1}{1.8}(0.2)+\frac{1}{2}(0.2)$$

$$=\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}=\frac{13}{42}+\frac{17}{72}+\frac{1}{10}$$

$$L(P,f)=\frac{936+714}{3024}+\frac{1}{10}=\frac{1650}{3024}+0.1=0.5456+0.1=0.6456$$

4\. Find upper and lower Riemann sums of $f(x)=2x-1$ on $[0,1]$ for the partition $P=\{0,\frac{1}{3},\frac{2}{3},1\}$.

Sol:- Sub intervals are $I_1=[0,\frac{1}{3}]$; $I_2=[\frac{1}{3},\frac{2}{3}]$; $I_3=[\frac{2}{3},1]$.

Length of each subinterval is $\delta_1=\delta_2=\delta_3=\frac{1}{3}$

Supremum and Infimum of $I_1 \Rightarrow M_1 = -\frac{1}{3}$; $m_1 = -1$

$I_2 \Rightarrow M_2 = \frac{1}{3}$; $m_2 = -\frac{1}{3}$

$I_3 \Rightarrow M_3 = 1$; $m_3 = \frac{1}{3}$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = M_1\delta_1 + M_2\delta_2 + M_3\delta_3 = -\frac{1}{3}\left(\frac{1}{3}\right) = \frac{1}{3}\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right)$$

$$= -\frac{1}{9} + \frac{1}{9} + \frac{1}{3} = \frac{1}{3}$$

$$L(P,f) = \sum_{r=1}^{n} m_r \delta_r = m_1\delta_1 + m_2\delta_2 + m_3\delta_3 = -1\left(\frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{1}{3}\right)$$

$$= -\frac{1}{3} - \frac{1}{9} + \frac{1}{9} = -\frac{1}{3}$$

* NOTE:- A closed interval $[a,b]$ can be partition in infinitely many ways the set of all partions of $[a,b]$ is denoted by $\phi[a,b]$

$\Rightarrow$ If $f:[a,b] \to R$ is a bounded function and $P \in \phi[a,b]$ then $U(P,f) - L(P,f)$ is called the oscillatory sum of f corresponding to the partition P.

$\Rightarrow$ $U(P,f) - L(P,f)$ is denoted by $W(P,f)$ or $O(P,f)$.

$\Rightarrow$ $W(P,f) = U(P,f) - L(P,f) \geq 0$.

* Upper and Lower Riemann Integrals:-

Let $f:[a,b] \to R$ is a bounded function and $P = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$ be a partition of $[a,b]$.

* The Lower Riemann Integral of f on $[a,b]$ is defined as $Sup\{L(P,f) / P \in \phi[a,b]\}$ and is denoted by $\underline{\int_a^b} f(x).dx$

$$\underline{\int_a^b} f(x).dx = Sup\{L(P,f) / P \in \phi[a,b]\}$$

* The Upper Riemann Integral of f on $[a,b]$ is defined as Infimum of $\{U(P,f) / P \in \phi[a,b]\}$ and is denoted by $\overline{\int_a^b} f(x).dx$.

$$\overline{\int_a^b} f(x).dx = \text{Inf}\{U(P,f) / P \in \phi[a,b]\}$$

* Riemann Integral:-

Let $f:[a,b] \rightarrow R$ be a bounded function and $P = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$ be a partition of f on $[a,b]$.

If $\underline{\int_a^b} f(x).dx = \text{Sup}\{L(P,f) / P \in \phi[a,b]\}$ is equal to $\overline{\int_a^b} f(x).dx = \text{Inf}\{U(P,f) / P \in \phi[a,b]\}$ then we say that f is Riemann Integrable over $[a,b]$ and the common value of these integrals is called the Riemann Integral of f on $[a,b]$.

The Riemann Integral of f on $[a,b]$ is denoted by $\int_a^b f(x).dx$.

* NOTE:-

$\Rightarrow$ A bounded function f is Riemann Integrable on $[a,b]$ if and only if $\overline{\int_a^b} f(x).dx = \underline{\int_a^b} f(x).dx$

$\Rightarrow$ If $f:[a,b] \rightarrow R$ is a bounded function, then

$$\underline{\int_a^b} f(x).dx \leq \overline{\int_a^b} f(x).dx$$

$\Rightarrow$ $\overline{\int_a^b} f(x).dx = \lim_{n\to\infty} U(P,f)$ and $\Rightarrow$ $\underline{\int_a^b} f(x).dx = \lim_{n\to\infty} L(P,f)$

5. Show that a constant function is Riemann Integrable.

Sol:- Let $f(x) = k \ \forall x \in [a,b]$ where k is a real number

Here f is a constant function.

clearly f is bounded on $[a,b]$ and $\operatorname{Inf} f = k$; $\operatorname{Sup} f = k$.

Let $P = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$ be a patition on $[a,b]$.

m_r, M_r are Infimum and Supremum of f on $I_r = [x_{r-1}, x_r]$

Since $f(x) = k \ \forall \ x \in [a,b] \Rightarrow m_r = k$; $M_r = k$

$$L(P,f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} k.\delta_r = k.\sum_{r=1}^{n} \delta_r = k(b-a)$$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} k.\delta_r = k.\sum_{r=1}^{n} \delta_r = k(b-a)$$

$$\underline{\int_a^b} f(x).dx = \operatorname{Sup}\{L(P,f) / P \in \phi[a,b]\}$$

$$= k(b-a)$$

$$\overline{\int_a^b} f(x).dx = \operatorname{Inf}\{U(P,f) / P \in \phi[a,b]\}$$

$$= k(b-a)$$

Here $\underline{\int_a^b} f(x).dx = \overline{\int_a^b} f(x).dx$

$\therefore f$ is R-Integrable on $[a,b]$.

i.e, A constant function is Integrable on $[a,b]$

6. The function $f(x)=1$ when $x \in \mathbb{Q}$ and $f(x)=-1$ when $x \in \mathbb{R}-\mathbb{Q}$. Prove that f is not Riemann Integrable on $[a,b]$.

Sol: By the definition of f, $-1 \le f(x) \le 1 \;\; \forall x \in [a,b]$

clearly f is a bounded function.

$\text{Inf } f = -1 \;;\; \text{Sup } f = 1$

Let $P = \{a = x_0, x_1, x_2, \text{-------} x_n = b\}$ be a ~~function~~ partition on $[a,b]$ & m_r, M_r are Inf & Sup of f on $I_r = [x_{r-1}, x_r]$.

$$\Rightarrow \quad m_r = -1 \;;\; M_r = 1$$

$$L(P,f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} (-1).\delta_r = (-1)\sum_{r=1}^{n} \delta_r = (-1)(b-a) = a-b$$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} (1).\delta_r = 1 \sum_{r=1}^{n} \delta_r = b-a$$

$$\underline{\int_a^b} f(x).dx = \text{Sup}\{L(P,f) / P \in \phi[a,b]\}$$

$$= a-b$$

$$\overline{\int_a^b} f(x).dx = \text{Inf}\{U(P,f) / P \in \phi[a,b]\}$$

$$= b-a$$

Here $\underline{\int_a^b} f(x).dx \neq \overline{\int_a^b} f(x).dx$

$\therefore f$ is not R-integrable on $[a,b]$.

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* **Theorem-1:-** If $f \in R[a,b]$ and m, M are Infimum and Supremum of f on $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x).dx \leq M(b-a).$$

proof:-

Since $f \in R[a,b]$,

$$\Rightarrow \underline{\int_a^b} f(x).dx = \overline{\int_a^b} f(x).dx = \int_a^b f(x).dx$$

Let m, M are Infimum and Supremum of f on $[a,b]$.

Let $P = \{a = x_0, x_1, x_2, \ldots \ldots x_n = b\}$ be a partition on $[a,b]$.

And m_r, M_r are Infimum and Supremum of f on $I_r = [x_{r-1}, x_r]$.

So, we have $m \leq m_r \leq M_r \leq M$. $\forall\, r = 1, 2, \ldots, n$

$$\Rightarrow m.\delta_r \leq m_r.\delta_r \leq M_r.\delta_r \leq M.\delta_r \text{ for } r = 1, 2, 3, \ldots n$$

$$\Rightarrow \sum_{r=1}^{n} m\delta_r \leq \sum_{r=1}^{n} m_r\delta_r \leq \sum_{r=1}^{n} M_r\delta_r \leq \sum_{r=1}^{n} M.\delta_r, \quad r = 1, 2, \ldots n$$

$$\Rightarrow m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a) \rightarrow (A)$$

~~Since~~, The above inequality is true for each $P \in \wp[a,b]$

Since, $\underline{\int_a^b} f(x) dx = \text{Sup}\{L(P,f) / P \in \wp[a,b]\} \Rightarrow \underline{\int_a^b} f(x).dx \geq L(P,f)$

$\overline{\int_a^b} f(x) dx = \text{Inf}\{U(P,f) / P \in \wp[a,b]\} \Rightarrow \overline{\int_a^b} f(x).dx \leq U(P,f)$

from (A), $\Rightarrow m(b-a) \leq L(P,f) \leq \underline{\int_a^b} f(x).dx$ &

$$\overline{\int_a^b} f(x).dx \leq U(P,f) \leq M(b-a)$$

$\Rightarrow m(b-a) \leq L(P,f) \leq \int_a^b f(x).dx \leq U(P,f) \leq M(b-a)$ [$\because f \in R[a,b]$]

$\Rightarrow m(b-a) \leq \int_a^b f(x).dx \leq M(b-a)$

Hence Proved.

Darboux's Theorem:-

If $f:[a,b] \to R$ is a bounded function, then for each $\varepsilon > 0$, $\exists\ \delta > 0 \ni$

i) $U(P,f) \leq \overline{\int_a^b} f(x).dx + \varepsilon$

ii) $L(P,f) \geq \underline{\int_a^b} f(x).dx - \varepsilon$

for each $P \in \phi[a,b]$ with $\|P\| < \delta$.

* **Necessary and sufficient condition for integrability:-**

A bounded function $f:[a,b] \to R$ is Riemann integrable on $[a,b]$ if and only if for each $\varepsilon > 0$, $\exists$ a Partition P on $[a,b]$ such that $0 \leq U(P,f) - L(P,f) < \varepsilon$.

proof:-

NECESSARY part:-

Let f be Riemann integrable on $[a,b]$

$\therefore \underline{\int_a^b} f(x).dx = \overline{\int_a^b} f(x).dx = \int_a^b f(x).dx$ — ①

Let $\varepsilon > 0$, by Darboux's theorem, $\exists\ \delta > 0$ such that

$U(P,f) < \overline{\int_a^b} f(x).dx + \frac{\varepsilon}{2}$ — ② and $L(P,f) > \underline{\int_a^b} f(x).dx - \frac{\varepsilon}{2}$ — ③

for each $P \in \phi[a,b]$ with $\|P\| < \delta$.

From ① & ②, $\Rightarrow U(P,f) < \int_a^b f(x).dx + \frac{\varepsilon}{2}$ — ④

From ① & ③, $\Rightarrow L(P,f) > \int_a^b f(x).dx - \frac{\varepsilon}{2}$ — ⑤

$$\Rightarrow \int_a^b f(x).dx < L(P,f) + \frac{\varepsilon}{2} \quad \text{— ⑤}$$

from ④ & ⑤,

$$U(P,f) < \left(L(P,f) + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2}$$

$$\Rightarrow U(P,f) < L(P,f) + \varepsilon$$

$$\Rightarrow U(P,f) - L(P,f) < \varepsilon. \quad \text{Also } U(P,f) - L(P,f) \geq 0$$

$$\therefore 0 \leq U(P,f) - L(P,f) < \varepsilon$$

<u>SUFFICIENT</u> <u>part</u>:-

Let for each $\varepsilon > 0$, $\exists$ a partition $P \in \wp[a,b]$ such that $0 \leq U(P,f) - L(P,f) < \varepsilon$.

we have $\underline{\int_a^b} f(x)\, dx = \text{Sup}\{L(P,f) / P \in \wp[a,b]\}$

$$\Rightarrow \underline{\int_a^b} f(x).dx \geq L(P,f) \rightarrow ⑥$$

and

$$\overline{\int_a^b} f(x).dx = \text{Inf}\{U(P,f) / P \in \wp[a,b]\}$$

$$\Rightarrow \overline{\int_a^b} f(x).dx \leq U(P,f) \rightarrow ⑦$$

from ⑥, $\Rightarrow -\underline{\int_a^b} f(x).dx \leq -L(P,f) \rightarrow ⑧$

from ⑦ & ⑧

$$\overline{\int_a^b} f(x).dx - \underline{\int_a^b} f(x).dx \leq U(P,f) - L(P,f) < \varepsilon$$

Also we have,

$$\overline{\int_a^b} f(x).dx - \underline{\int_a^b} f(x).dx \geq 0$$

for $\varepsilon > 0$,

$$\Rightarrow -\varepsilon < 0 \leq \overline{\int_a^b} f(x).dx - \underline{\int_a^b} f(x).dx < \varepsilon$$

$$\Rightarrow -\varepsilon < \overline{\int_a^b} f(x).dx - \underline{\int_a^b} f(x).dx < \varepsilon$$

$$\Rightarrow \left| \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x).dx \right| < \varepsilon$$

$$\Rightarrow \lim \left(\overline{\int_a^b} f(x).dx - \underline{\int_a^b} f(x).dx \right) = 0$$

$$\Rightarrow \overline{\int_a^b} f(x).dx - \underline{\int_a^b} f(x).dx = 0$$

$$\Rightarrow \overline{\int_a^b} f(x).dx = \underline{\int_a^b} f(x).dx$$

$\therefore$ f is Riemann Integrable on $[a,b]$.

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7. Show that $f(x) = 3x+1$ is integrable on $[1,2]$ and $\int_1^2 f(x).dx = \frac{11}{2}$.

Sol:- consider $f(x) = 3x+1$

since, f is bounded on $[1,2]$

consider a partition $P = \{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \ldots\ldots, 1+\frac{n}{n}\}$

Let the subinterval $I_r = \left[1+\frac{(r-1)}{n}, 1+\frac{r}{n}\right]$

and length of each subinterval $= \frac{1}{n}$

Also Inf & Sup of f in I_r are

$$m_r = 3\left(1+\frac{r-1}{n}\right)+1 = 4 + \frac{3(r-1)}{n}$$

$$M_r = 3\left(1+\frac{r}{n}\right)+1 = 4+\frac{3r}{n}$$

$$U(P,f) = \sum_{r=1}^{n} M_r . \delta_r$$

$$= \sum_{r=1}^{n} \left(4+\frac{3r}{n}\right)\frac{1}{n}$$

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$$U(P,f) = \sum_{r=1}^{n}\left(\frac{4}{n}+\frac{3r}{n^2}\right) = \sum_{r=1}^{n}\frac{4}{n} + \sum_{r=1}^{n}\frac{3r}{n^2}$$

$$= \frac{4}{n}\sum_{r=1}^{n}(1) + \frac{3}{n^2}\sum_{r=1}^{n} r = \frac{4}{n}(n) + \frac{3}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= 4 + \frac{3}{n}\left(\frac{n+1}{2}\right) = 4 + \frac{3}{2}\left(1+\frac{1}{n}\right) = \frac{11}{2} + \frac{3}{2n}$$

$$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$$

$$= \sum_{r=1}^{n}\left[4+3\left(\frac{r-1}{n}\right)\right]\frac{1}{n} = \sum_{r=1}^{n}\left(\frac{4}{n}+3\left(\frac{r-1}{n^2}\right)\right)$$

$$= \sum_{r=1}^{n}\frac{4}{n} + \sum_{r=1}^{n}\frac{3(r-1)}{n^2} = \frac{4}{n}\sum_{r=1}^{n}(1) + \frac{3}{n^2}\sum_{r=1}^{n}(r-1)$$

$$= \frac{4}{n}(n) + \frac{3}{n^2}\left(\frac{(n-1)(n)}{2}\right) = 4 + \frac{3}{2}\left(1-\frac{1}{n}\right) = \frac{11}{2} - \frac{3}{2n}$$

we have,

$$\overline{\int_{a}^{b}} f(x).dx = \lim_{n\to\infty} U(P,f)$$

$$\overline{\int_{1}^{2}} f(x).dx = \lim_{n\to\infty} \frac{11}{2} + \frac{3}{2n} = \frac{11}{2} + 0 = \frac{11}{2}$$

$$\underline{\int_{a}^{b}} f(x).dx = \lim_{n\to\infty} L(P,f)$$

$$\underline{\int_{1}^{2}} f(x).dx = \lim_{n\to\infty} \frac{11}{2} - \frac{3}{2n} = \frac{11}{2} - 0 = \frac{11}{2}$$

Here, $\overline{\int_{1}^{2}} f(x).dx = \underline{\int_{1}^{2}} f(x).dx$

$$\therefore \int_{1}^{2} f(x).dx = \frac{11}{2}$$

And $f(x)$ is integrable on $[1,2]$.

3. Prove that $f(x) = x^2$ is integrable on $[0,a]$ and $\int_0^a x^2 . dx = \frac{a^3}{3}$.

Sol:- Given that, $f(x) = x^2$

since, f is bounded on $[0, a]$

consider a partition $P = \{0, \frac{a}{n}, \frac{2a}{n}, \frac{3a}{n}, \ldots, \frac{na}{n}\}$

SubInterval $I_r = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$

and length of each subinterval $(\delta_r) = \frac{a}{n}$

consider Sup and Inf of subinterval I_r,

$$M_r = \left(\frac{ra}{n}\right)^2 = \frac{r^2a^2}{n^2} \quad ; \quad m_r = \left(\frac{(r-1)a}{n}\right)^2 = \frac{(r-1)^2.a^2}{n^2}$$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n}\left(\frac{r^2a^2}{n^2}\right)\frac{a}{n} = \frac{a^3}{n^3}\sum_{r=1}^{n} r^2$$

$$= \frac{a^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{a^3}{6}\left[\frac{n(n+1)(2n+1)}{n.n.n}\right]$$

$$= \frac{a^3}{6}(1+1/n)\left(2+\frac{1}{n}\right)$$

$$L(P,f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n}\left(\frac{(r-1)^2a^2}{n^2}\right)\frac{a}{n} = \frac{a^3}{n^3}\sum_{r=1}^{n}(r-1)^2$$

$$= \frac{a^3}{n^3}\left[\frac{(n-1)(n-1+1)(2(n-1)+1)}{6}\right] = \frac{a^3}{6}\left[\frac{(n-1)n(2n-1)}{n.n.n}\right]$$

$$= \frac{a^3}{6}\left[\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right]$$

$$\int_{\bar{a}}^{b} f(x).dx = \lim_{n\to\infty} L(P,f)$$

$$\int_{\bar{0}}^{a} f(x)dx = \lim_{n\to\infty}\left[\frac{a^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right] = \frac{a^3}{6}(1).(2) = \frac{a^3}{3}$$

$$\int_0^{\bar{a}} f(x).dx = \lim_{n\to\infty}\left[\frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = \frac{a^3}{6}(1).(2) = \frac{a^3}{3}$$

Here, $\int_{\bar{0}}^{a} f(x).dx = \int_0^{\bar{a}} f(x).dx$

$$\therefore \int_0^a f(x).dx = \frac{a^3}{3}$$

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9. Prove that $f(x) = \sin x$ is integrable on $\left[0, \frac{\pi}{2}\right]$ and

$$\int_0^{\pi/2} \sin x . dx = 1.$$

Sol: Consider $f(x) = \sin x$.

Since, f is bounded on $\left[0, \frac{\pi}{2}\right]$.

consider the partition,

$$P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \ldots\ldots, \frac{n\pi}{2n}\right\}$$

Subinterval $I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right]$

and Length of each sub interval $(\delta_r) = \frac{\pi}{2n}$

consider Sup and Inf of subinterval I_r are

$$M_r = \sin\left(\frac{r\pi}{2n}\right) \quad ; \quad m_r = \sin\left(\frac{(r-1)\pi}{2n}\right)$$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} \left[\sin\frac{r\pi}{2n}\right] \cdot \frac{\pi}{2n}$$

$$= \frac{\pi}{2n} \cdot \sum_{r=1}^{n} \sin\frac{r\pi}{2n}$$

$$= \frac{\pi}{2n}\left[\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \ldots\ldots + \sin\frac{n\pi}{2n}\right]$$

$$= \frac{\pi}{2n}\left[\frac{\sin\left(\frac{\pi}{2n} + \left(\frac{n-1}{2}\right)\cdot\frac{\pi}{2n}\right) \cdot \sin\frac{n}{2}\left(\frac{\pi}{2n}\right)}{\sin\frac{\pi}{4n}}\right]$$

$$= \frac{\pi}{2\sqrt{2}n}\left[\sin\left(\frac{\pi}{2n} + \frac{n\pi}{4n} - \frac{\pi}{4n}\right)\right] \Big/ \sin\frac{\pi}{4n}$$

$$= \frac{\pi}{2\sqrt{2}n} \cdot \sin\left[\frac{\pi}{4} + \frac{\pi}{4n}\right] \Big/ \sin\frac{\pi}{4n}$$

$$= \frac{\pi}{2\sqrt{2}n} \cdot \left[\sin\frac{\pi}{4} \cdot \cos\frac{\pi}{4n} + \cos\frac{\pi}{4} \cdot \sin\frac{\pi}{4n}\right] \Big/ \sin\frac{\pi}{4n}$$

$$= \frac{\pi}{2\sqrt{2}n} \cdot \frac{1}{\sqrt{2}}\left[\cos\frac{\pi}{4n} + \sin\frac{\pi}{4n}\right] \Big/ \sin\frac{\pi}{4n}$$

$$U(P,f) = \frac{\pi}{4n} \cdot \left[\cot\frac{\pi}{4n} + 1\right]$$

similarly, $L(P,f) = \frac{\pi}{4n}\left[\cot\frac{\pi}{4n} - 1\right]$

$$\int_0^{\pi/2} \sin x \, dx = \lim_{n\to\infty} U(P,f) = \lim_{n\to\infty} \frac{\pi}{4n}\left[\cot\frac{\pi}{4n} + 1\right]$$

$$= \lim_{n\to\infty}\left[\frac{\frac{\pi}{4n}}{\tan\frac{\pi}{4n}} + \frac{\pi}{4n}\right] = \lim_{n\to\infty} \frac{\pi/4n}{\tan\frac{\pi}{4n}} + \lim_{n\to\infty}\frac{\pi}{4n}$$

$$= 1 + 0 = 1$$

similarly, $\int_0^{\pi/2} \sin x \, dx = 1$

Another definition of Riemann Integral :-

Let $f : [a,b] \to R$ be a function and $P = \{a = x_0, x_1, \cdots x_n = b\}$ be a partition of $[a,b]$. Let $\{\xi_1, \xi_2, \cdots \xi_n\} \subset [a,b]$ be such that $x_{r-1} \le \xi_r \le x_r$ for $r = 1, 2, \cdots n$. The function f is said to be Riemann integrable over $[a,b]$, if to each $\varepsilon > 0$ $\exists \delta > 0$ $\ni$ a number I such that $\left| \sum_{r=1}^{n} f(\xi_r)\delta_r - I \right| < \varepsilon$ for $P \in \phi[a,b]$ with $\|P\| < \delta$ & $\xi_r \in [x_{r-1}, x_r]$. The number I is the Riemann Integral of f over $[a,b]$. ie: $\int_a^b f(x)\,dx$

Note: f is integrable on $[a,b] \Rightarrow \int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$

$\Rightarrow$ A function $f : [a,b] \to R$ is integrable on $[a,b]$ if
(i) f is continuous on $[a,b]$
(ii) f is monotonic on $[a,b]$
(iii) f is bounded on $[a,b]$ with a finite number of points of discontinuity
(iv) f is bounded on $[a,b]$ & the set of its points of discontinuity has a finite number of limit points.

Theorem :- If $f : [a,b] \to R$ is continuous on $[a,b]$, then f is integrable on $[a,b]$

Proof: Since f is continuous on $[a,b] \Rightarrow f$ is bounded on $[a,b]$

f is continuous on $[a,b] \Rightarrow$ By Borel's theorem for each $\varepsilon > 0$ $\exists$ a partition $P = \{x_0, x_1, \cdots x_n\}$ of $[a,b]$ such that

$|f(y_r) - f(z_r)| < \frac{\varepsilon}{b-a}$ for $y_r, z_r \in I_r$, $r = 1, 2, \cdots n$

For, the partition P, let m_r, M_r be the Infimum & Supremum of f in I_r.

f is continuous on $I_r \Rightarrow$ There exist $\alpha_r, \beta_r \in I_r$ such that

$m_r = f(\alpha_r)$, $M_r = f(\beta_r)$

$\therefore M_r - m_r = |f(\alpha_r) - f(\beta_r)| < \frac{\varepsilon}{b-a}$ for $r = 1, 2, \cdots n$

$$\therefore W(P,f) = U(P,f) - L(P,f) = \sum_{r=1}^{n} (M_r - m_r)\delta_r$$

$$< \sum_{r=1}^{n} \left(\frac{\varepsilon}{b-a}\right)\delta_r$$

$$W(P,f) = U(P,f) - L(P,f) < \frac{\varepsilon}{b-a}\sum_{r=1}^{n}\delta_r$$
$$= \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$

$\therefore$ Hence for each $\varepsilon > 0$ $\exists$ $P \in \phi[a,b]$ $\ni$ $0 \le W(P,f) < \varepsilon$

$\therefore$ f is integrable on $[a,b]$.

Note: There exist a function which are integrable but not continuous. So continuity is a sufficient condition but not necessary

Ex: $f(x) = e^x$ is continuous on $[a,b] \subset R$ & e^x is integrable on $[a,b]$.

Theorem: If $f:[a,b] \to R$ is monotonic on $[a,b]$, then f is integrable on $[a,b]$.

Proof: Let f be increasing on $[a,b]$. Then $f(a) \le f(x) \le f(b)$ $\forall$ $x \in [a,b]$.

$\therefore$ f is bounded on $[a,b]$ & $\inf f = f(a)$ & $\sup f = f(b)$

Let $\varepsilon > 0$ and $P = \{a = x_0, x_1, \ldots x_n = b\}$ be a partition of $[a,b]$ such that $\delta_r < \frac{\varepsilon}{f(b)-f(a)+1}$ for $r = 1, 2, \cdots n$

Let m_r, M_r be the inf & sup of f on I_r, then

$m_r = f(x_{r-1})$, $M_r = f(x_r)$

$$W(P,f) = \sum_{r=1}^{n}(M_r - m_r)\delta_r = \sum_{r=1}^{n}\left[f(x_r) - f(x_{r-1})\right]\delta_r$$
$$< \sum_{r=1}^{n}\left[f(x_r) - f(x_{r-1})\right]\frac{\varepsilon}{f(b)+f(a)+1}$$
$$< \frac{\varepsilon}{f(b)-f(a)+1}\sum_{r=1}^{n}\left[f(x_r) - f(x_{r-1})\right]$$
$$< \frac{\varepsilon}{f(b)-f(a)+1}\left[f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1})\right]$$
$$< \frac{\varepsilon}{f(b)-f(a)+1}\left[f(x_1) - f(x_0)\right]$$
$$< \frac{\varepsilon(f(b)-f(a))}{f(b)-f(a)+1} < \varepsilon$$

for each $\varepsilon > 0$ $\exists$ a partition $P \in \wp[a,b]$ such that $0 \le W(P,f) < \varepsilon$

$\therefore f$ is integrable on $[a,b]$.

Ily we can prove that f is integrable when f is decreasing on $[a,b]$

Ex: ① $f(x) = \frac{1}{x}$ is decreasing on $[1,2]$, so $\frac{1}{x}$ is integrable on $[1,2]$

② $f(x) = x^2$ is increasing on $[a,b]$ when $b > a > 0$ and so x^2 is integrable on $[a,b]$

* Show that $f(x) = [x]$ is integrable on $[0,3]$ & find $\int_0^3 [x]\,dx$

Sol: We know that f is continuous on $[0,3]$ except at $1,2,3$.

ie; the set of points of discontinuity of f on $[0,3]$ is $\{1,2,3\}$, a finite set

$\therefore f$ is Riemann integrable on $[0,3]$

$$\text{Now, } \int_0^3 f(x)\,dx = \int_0^3 [x]\,dx = \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx$$

$$= 0 + [x]_1^2 + [2x]_2^3$$

$$= (2-1) + (6-4) = 3$$

$$\therefore \int_0^3 [x]\,dx = 3$$

* Note: If the set of points of discontinuity of a bounded function $f : [a,b] \to \mathbb{R}$ has a finite number of limit points then f is integrable on $[a,b]$.

* Problem:- If $f(x) = \frac{1}{2^n}$ when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$; $(n = 0,1,2,\ldots)$ & $f(0) = 0$ then prove that f is integrable on $[0,1]$.

Sol: Given function is $f(x) = \frac{1}{2^n}$ when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$, where $n = 0,1,2,\ldots$ & $f(0) = 0$

ie; for $n=0$, $f(x) = 1$ when $\frac{1}{2^{0+1}} < x \le \frac{1}{2^0} = \frac{1}{2} < x \le 1$

for $n=1$; $f(x) = \frac{1}{2}$ when $\frac{1}{2^2} < x \le \frac{1}{2}$

$\vdots$

Now, we check the continuity at $x = \frac{1}{2^n}$

$$\lim_{x\to \frac{1}{2^n}^-} f(x) = \lim_{x\to \frac{1}{2^n}} \frac{1}{2^n} = \frac{1}{2^n} \qquad \left(\because f(x) = \frac{1}{2^n} \text{ when } \frac{1}{2^{n+1}}\right.$$

$$\lim_{x\to \frac{1}{2^n}^+} f(x) = \lim_{x\to \frac{1}{2^n}} \frac{1}{2^{n-1}} = \frac{1}{2^{n-1}}$$

$\because \lim_{x\to \frac{1}{2^n}^-} f(x) \neq \lim_{x\to \frac{1}{2^n}^+} f(x) \Rightarrow f$ is not continuous at $x = \frac{1}{2^n}$ for $n = 0,1,2,$

The set of Points of discontinuity of f on $[0,1]$ is

$$\left\{ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n}, \dots \right\}$$

this set has a unique limit point '0'

$\therefore f$ is Riemann integrable on $[0,1]$

Now, we find the integrable at this function

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_{1/2^n}^1 f(x)\,dx$$

consider $\displaystyle\int_{1/2^n}^1 f(x)\,dx = \int_{1/2^n}^{1/2^{n-1}} \frac{1}{2^{n-1}}\,dx + \int_{1/2^{n-1}}^{1/2^{n-2}} \frac{1}{2^{n-2}}\,dx + \dots + \dots + \frac{1}{2^2}\int_{1/2}^{1/2} dx + \int_{1/2}^{1} dx$

$$\Rightarrow \int_{1/2^n}^1 f(x)\,dx = \frac{1}{2^{n-1}}[x]_{1/2^n}^{1/2^{n-1}} + \frac{1}{2^{n-2}}[x]_{1/2^{n-1}}^{1/2^{n-2}} + \dots + \dots + \frac{1}{2}[x]_{1/2^2}^{1/2} + 1\cdot[x]_{1/2}^{1}$$

$$\Rightarrow \int_{1/2^n}^1 f(x)\,dx = \frac{1}{2^{n-1}}\left[\frac{1}{2^{n-1}} - \frac{1}{2^n}\right] + \frac{1}{2^{n-2}}\left[\frac{1}{2^{n-2}} - \frac{1}{2^{n-1}}\right] + \dots + \dots + \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2^2}\right] + \left[1 - \frac{1}{2}\right]$$

$$\Rightarrow \int_{1/2^n}^1 f(x)\,dx = \frac{1}{2^{n-1}}\left[\frac{1}{2^n}\right] + \frac{1}{2^{n-2}}\left[\frac{1}{2^{n-1}}\right] + \dots + \frac{1}{2}\left[\frac{1}{2^2}\right] + \frac{1}{2}$$

$$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \dots + \frac{1}{(2^{n-2})^2} + \frac{1}{(2^{n-1})^2}\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{4} + \dots + \frac{1}{4^{n-2}} + \frac{1}{4^{n-1}}\right]$$

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$$= \frac{1}{2}\left[\frac{1\left(1-\left(\frac{1}{4}\right)^n\right)}{1-\frac{1}{4}}\right]$$

$$= \frac{1}{2}\cdot\frac{4}{3}\left[1-\frac{1}{4^n}\right]$$

$$= \frac{2}{3}\left[1-\frac{1}{4^n}\right]$$

$$\therefore \int_0^1 f(x)\,dx = \lim_{n\to\infty}\int_{\frac{1}{2^n}}^1 f(x)\,dx = \lim_{n\to\infty}\frac{2}{3}\left[1-\frac{1}{4^n}\right]$$

$$= \frac{2}{3} - \lim_{n\to\infty}\frac{2}{3\cdot 4^n}$$

$$\int_0^1 f(x)\,dx = \frac{2}{3}$$

Properties of Integrable functions :-

(1) $\int_a^a f(x)\,dx = 0$

(2) If $f \in R[b,a]$, where $b<a$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(3) If $f \in R[a,b]$ & $K \in R$, then $Kf \in R[a,b]$ & $\int_a^b (Kf)(x)\,dx = K\int_a^b f(x)\,dx$

(4) If $f,g \in R[a,b]$, then $f+g \in R[a,b]$ and $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(5) If $f \in R[a,b]$, then $f^2 \in R[a,b]$

(6) If $f,g \in R[a,b]$, then $f.g \in R[a,b]$.

(7)

Theorem: If $f \in R([a,b])$ then $|f| \in R[a,b]$

Proof: $f \in R[a,b] \Rightarrow f$ is bounded on $[a,b]$ &
for each $\epsilon>0$ $\exists$ a partition $P=\{a=x_0, x_1, \ldots x_n=b\}$ of $[a,b]$
such that $U(P,f) - L(P,f) < \epsilon$

f is bounded on $[a,b] \Rightarrow |f(x)| \le K$ for some $x \in R^+$
$\Rightarrow |f(x)| \le K$
$\Rightarrow |f|$ is bounded

let m_r, M_r & m_r', M_r' be Supremum & Infimum of f & $|f|$ respectively on I_r.

for each $\alpha, \beta \in I_r$, $\left| |f(\alpha)| - |f(\beta)| \right| \leq |f(\alpha) - f(\beta)|$

$\therefore M_r' - m_r' \leq M_r - m_r$ for $r = 1, 2, \cdots n$

$$\Rightarrow \sum_{r=1}^{n} M_r' \delta_r - \sum_{r=1}^{n} m_r' \delta_r \leq \sum_{r=1}^{n} M_r \delta_r - \sum_{r=1}^{n} m_r \delta_r$$

$$\Rightarrow U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f) < \varepsilon$$

By N&S condition, $|f|$ is Riemann Integrable on $[a, b]$.

Converse of this theorem is not true if $|f|$ is Riemann Integrable then f need not be Riemann Integrable.

$$f(x) = \begin{cases} 1 & \text{for } x \in Q \\ -1 & \text{for } x \in R - Q \end{cases}$$

$|f|$ is defined as $|f|x = |f(x)| = 1 \ \forall x \in R$

$\therefore |f|$ is constant, $|f|$ is Riemann Integrable on $[a, b] \subseteq R$

let $P = \{a = x_0, x_1, \cdots x_n = b\}$ be a partition of $[a, b] \subseteq R$

let M_r, m_r be Sup & Inf of f on I_r.

$\Rightarrow M_r = 1, m_r = -1$ for $r = 1, 2, \cdots n$

$$U(P, f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} \delta_n = x_n - x_0 = b - a, \text{ a constant}$$

$$\therefore \overline{\int_a^b} f(x)\, dx = \mathrm{Inf}\{U(P, f) / P \in \phi[a, b]\} = b - a$$

$$L(P, f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} (-1)\delta_r = -1[x_1 - x_0 + x_2 - x_1 + \cdots + x_n - x_{n-1}]$$

$$= -1[x_n - x_0]$$

$$= x_0 - x_n$$

$$\therefore \underline{\int_a^b} f(x)\, dx = \mathrm{Sup}\{L(P, f) / P \in \phi[a, b]\} = a - b$$

$$= a - b$$

$\therefore \overline{\int_a^b} f(x)\, dx \neq \underline{\int_a^b} f(x)\, dx$, f is not Riemann integrable on $[a, b]$.

Theorem: If $f \in R([a, b])$ and $c \in [a, b]$ then $f \in R([a, c])$ & $f \in R([c, b])$ and also $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

Proof: Let f be Riemann Integrable on $[a, b]$

$\Rightarrow f$ is bounded on $[a, b]$

$\Rightarrow f$ is bounded on $[a, c]$ & $[c, b]$

$P \in \wp([a,b])$ & $Q = P \cup \{c\}$

$\Rightarrow P \subseteq Q$

$\Rightarrow U(Q,f) \leq U(P,f)$ & $L(P,f) \leq L(Q,f)$

we take $Q_1 = Q \cap [a,c]$ & $Q_2 = Q \cap [c,b]$ Then Q_1 & Q_2 are Partition of $[a,c]$ & $[c,b]$ respectively.

So that $Q_1 \cup Q_2 = Q_1$ & $Q_1 \cap Q_2 = \{c\}$

$$\therefore L(P,f) \leq L(Q,f) = L(Q_1,f) + L(Q_2,f)$$

$$\leq \underline{\int_a^c} f(x)\,dx + \underline{\int_c^b} f(x)\,dx$$

$$\leq U(Q_1,f) + U(Q_2,f) = U(Q,f) \leq U(P,f)$$

This is true for every $P \in \wp([a,b])$

$$\therefore \underline{\int_a^b} f(x)\,dx \leq \underline{\int_a^c} f(x)\,dx + \underline{\int_c^b} f(x)\,dx \leq \overline{\int_a^c} f(x)\,dx + \overline{\int_c^b} f(x)\,dx$$

$$\leq \overline{\int_a^b} f(x)\,dx$$

$$\therefore f \in R[a,b];\ \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

$$\therefore \underline{\int_a^c} f(x)\,dx + \underline{\int_c^b} f(x)\,dx = \overline{\int_a^c} f(x)\,dx + \overline{\int_c^b} f(x)\,dx \quad \text{— (1)}$$

$$\Rightarrow \left[\overline{\int_a^c} f(x)\,dx - \underline{\int_a^c} f(x)\,dx\right] + \left[\overline{\int_c^b} f(x)\,dx - \underline{\int_c^b} f(x)\,dx\right] = 0$$

Since both terms are non-negative

$$\overline{\int_a^c} f(x)\,dx - \underline{\int_a^c} f(x)\,dx = 0\ ; \quad \overline{\int_c^b} f(x)\,dx - \underline{\int_c^b} f(x)\,dx = 0$$

$$\Rightarrow \overline{\int_a^c} f(x)\,dx = \underline{\int_a^c} f(x)\,dx \ \ \& \ \ \overline{\int_c^b} f(x)\,dx = \underline{\int_c^b} f(x)\,dx$$

$\Rightarrow f \in R[a,c]$ & $f \in R[c,b]$

also $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Integral function of function f :-

let $f \in R[a,b]$. then for each $t \in [a,b]$, $[a,t] \subset [a,b]$ hence $f \in R[a,t]$. therefore $\int_a^t f(x)dx$ is well-defined. the function $\phi(t) = \int_a^t f(x)dx$, $t \in [a,b]$ is called the Integral function of f. ϕ is also called Indefinite Integral f on $[a,b]$.

Theorem:- If $f \in R[a,b]$ then $\phi(t) = \int_a^t f(x)dx$, $t \in [a,b]$ is Continuous on $[a,b]$.

Proof: $f \in R[a,b] \Rightarrow f$ is bounded on $[a,b]$

$\Rightarrow$ There exists $K \in R^+ \ni |f(x)| \leq K \ \forall x \in [a,b]$

let $c \in [a,b]$ and $\varepsilon > 0$ be given. then ~~$\phi(c+h) - \phi(c) =$~~

$$\begin{aligned}
\phi(c+h) - \phi(c) &= \left| \int_a^{c+h} f(x)\,dx - \int_a^c f(x)dx \right| \\
&= \left| \int_a^c f(x)dx + \int_c^{c+h} f(x)dx - \int_a^c f(x)dx \right| \\
&= \left| \int_c^{c+h} f(x)dx \right| \\
&\leq K\,|c+h-c| = K|h| < \varepsilon \quad \text{if } |h| < \frac{\varepsilon}{K}
\end{aligned}$$

$\therefore$ for $\varepsilon > 0$, we can find a positive number $\delta \left(< \frac{\varepsilon}{K}\right)$ Such that $\phi(c+h) - \phi(c) < \varepsilon$ for $|h| < \delta$.

$\therefore \phi$ is Continuous at $c \in [a,b]$ which implies that ϕ is continuous on $[a,b]$

Theorem: If $f \in R[a,b]$ & f is Continuous at $c \in [a,b]$ then $\phi(t) = \int_a^t f(x)dx \ \forall t \in [a,b]$ is derivable at c & $\phi'(c) = f(c)$.

Proof: f is continuous at $c \in [a,b] \Rightarrow$ for each $\varepsilon > 0 \ \exists \delta > 0$ Such that $|f(x) - f(c)| < \varepsilon$ for $x \in [a,b]$ & $|x-c| < \delta$

take h so that $|h| < \delta$, $\phi(c+h) - \phi(c) = \int_a^{c+h} f(x)dx - \int_a^c f(x)dx$

$= \int_c^{c+h} f(x)dx$

and $\int_c^{c+h} f(c)\,dx = f(c)\int_c^{c+h} dx = f(c).h$

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$$\left|\frac{\phi(c+h)-\phi(c)}{h} - f(c)\right| = \left|\frac{1}{h}\int_c^{c+h} f(x)dx - \frac{1}{h}\int_c^{c+h} f(c)dx\right|$$

$$= \left|\frac{1}{h}\int_c^{c+h} (f(x)-f(c))dx\right|$$

$$\leq \frac{1}{|h|}\left|\int_c^{c+h} |f(x)-f(c)|\,dx\right|$$

$$< \frac{1}{|h|}\left|\int_c^{c+h} \varepsilon\,dx\right|$$

$$= \frac{1}{|h|}\cdot \varepsilon |h| = \varepsilon$$

$\therefore$ ϕ is derivable at c and $\phi'(c) = f(c)$

Primitive (or) Anti derivative of f :-

If $f \in R[a,b]$ & if there exist $\phi : [a,b] \to R$ such that $\phi'(x) = f(x)$ $\forall x \in [a,b]$, then ϕ is called a primitive (or) Anti derivative of f.

Fundamental theorem of Integral calculus :-

If $f \in (R[a,b])$ and ϕ is a primitive of f, then $\int_a^b f(x)dx = \phi(b) - \phi(a)$

Proof: Let $f \in R([a,b])$ then for $P = \{a = x_0, x_1, x_2, ---- x_n = b\}$ of $[a,b]$ and $x_{r-1} \leq \xi_r \leq x_r$, $r = 1,2,----n$

we have $\lim\limits_{\|P\| \to 0} \sum\limits_{r=1}^{n} f(\xi_r)\delta_r = \int_a^b f(x)\,dx$ ——— (1)

Let ϕ be a primitive of f, then $\phi'(x) = f(x)$ $\forall x \in [a,b]$

$\therefore$ ϕ is differentiable on $[a,b]$, ϕ is continuous on $[a,b]$

$\Rightarrow$ ϕ is continuous & differentiable on $[x_{r-1}, x_r]$ for $r = 1,2,---$

By Lagrange's ~~theorem~~ Mean value theorem,

$$\phi'(\xi_r) = \frac{\phi(x_r) - \phi(x_{r-1})}{x_r - x_{r-1}} \text{ for } \xi_r \in [x_{r-1}, x_r],\ r = 1,2,--r$$

now, $\sum\limits_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \sum\limits_{r=1}^{n} (x_r - x_{r-1}) f(\xi_r)$

$$\Rightarrow \phi(x_n) - \phi(x_0) = \sum_{r=1}^{n} f(\xi_r)\,\delta_r$$

$\Rightarrow \lim\limits_{\|P\| \to 0} [\phi(x_n) - \phi(x_0)] = \lim\limits_{\|P\| \to 0} \sum\limits_{r=1} f(\xi_r)\delta_r$

$\Rightarrow \phi(b) - \phi(a) = \int_a^b f(x)\,dx$ (from ①)

→ Show that $\int_0^1 x^4\,dx = \frac{1}{5}$

Sol: Let the function $f(x) = x^4$ is continuous on $\mathbb{R}$

$\Rightarrow f(x)$ is continuous on $[0,1]$

$\Rightarrow \int_0^1 x^4\,dx$ exists

Consider, $\phi(x) = \frac{x^5}{5}$ defined on $[0,1]$

Clearly, ϕ is derivable on $[0,1]$ & $\phi'(x) = x^4 = f(x)\ \forall\ x \in [0,1]$

$\therefore \phi$ is a primitive of f on $[0,1]$

$\therefore$ By fundamental theorem of Integral calculus

$\int_0^1 f(x)\,dx = \phi(1) - \phi(0) = \frac{1}{5} - 0 = \frac{1}{5}$

$\therefore \int_0^1 x^4\,dx = \frac{1}{5}$.

→ Show That $\int_a^b \cos x\,dx = \sin b - \sin a$

Let the function $f(x) = \cos x$ is Continuous on $\mathbb{R}$

$\Rightarrow f(x)$ is Continuous on $[a,b]$

$\Rightarrow \int_a^b f(x)\,dx$ exists $\Rightarrow \int_a^b \cos x\,dx$ exists

Consider, $\phi(x) = \sin x$ defined on $[a,b]$

Clearly, ϕ is derivable on $[a,b]$ & $\phi'(x) = \cos x = f(x)$

$\forall\ x \in [a,1]$

$\therefore \phi$ is primitive of f on $[0,1]$

By Fundamental theorem of Integral calculus

$\int_a^b f(x)\,dx = \phi(b) - \phi(a) = \sin b - \sin a$

→ Prove that $\int_a^b e^x\,dx = e^b - e^a$

Let the function $f(x) = e^x$ is continuous on $\mathbb{R}$

$\Rightarrow f(x)$ is continuous on $[a,b]$

$\Rightarrow \int_a^b f(x)\,dx$ exists $\Rightarrow \int_a^b e^x\,dx$ exists

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Consider $\phi(x) = e^x$ defined on $[a,b]$ (6)

Clearly, $\phi(x) = e^x$ derivable on $[a,b]$

$\phi'(x) = e^x = f(x) \; \forall \; x \in [a,b]$

$\therefore \phi$ is a primitive of f on $[a,b]$

$\Rightarrow$ By fundamental theorem $\int_a^b f(x)\,dx = \phi(b) - \phi(a) = e^b - e^a$

→ Evaluate $\int_0^{\pi/4} (\sec^4 x - \tan^4 x)\,dx$

Sol. The function $f(x) = \sec^4 x - \tan^4 x$

$$= (\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x)$$
$$= \sec^2 x + \tan^2 x$$
$$= \sec^2 x + (\sec^2 x - 1)$$
$$= 2\sec^2 x - 1$$

$\therefore f(x)$ is Continuous on $[0, \frac{\pi}{4}]$

$\Rightarrow \int_0^{\pi/4} (2\sec^2 x - 1)\,dx$ exists

Consider, $\phi(x) = 2\tan x - x$ defined on $[0, \frac{\pi}{4}]$

Clearly, ϕ is derivable on $[0, \frac{\pi}{4}]$ & $\phi'(x) = 2\sec^2 x - 1 = f(x)$

$\forall \; x \in [0, \frac{\pi}{4}]$

$\therefore$ By fundamental theorem $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$

$$\Rightarrow \int_a^b f(x)\,dx = \phi(b) - \phi(a)$$
$$= 2\tan\frac{\pi}{4} - \frac{\pi}{4} - [2\tan 0 - 0]$$
$$= 2(1) - \frac{\pi}{4} - 0$$
$$\Rightarrow \int_a^b f(x)\,dx = 2 - \frac{\pi}{4}$$

→ Prove that $f(x) = x[x]$ is Integrable on $[0,2]$ & $\int_0^2 x[x]\,dx = \frac{3}{2}$

Sol: We know that $f(x) = x[x]$

ie, discontinuous at $x = 1$

$\boxed{\text{for } x \to 1^- \; [0] = 0, \; x \to 1^+ \; [1] = 1 \; \neq}$

$\therefore f(x)$ is bounded and has finite no. of discontinuous points

$\Rightarrow f(x)$ is integrable on $[0,2]$

$$\Rightarrow \int_0^2 f(x)\,dx = \int_0^1 x[x]\,dx + \int_1^2 x[x]\,dx$$

$$= \int_0^1 x(0)\,dx + \int_1^2 x(1)\,dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

→ Evalate $\lim_{n\to\infty} \frac{1}{n}\left[e^{3/n} + e^{6/n} + \cdots + e^{3n/n}\right]$

Sol: $\lim_{n\to\infty} \frac{1}{n}\left[e^{3/n} + e^{6/n} + \cdots + e^{3n/n}\right]$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} e^{3r/n}$$

$$= \int_0^1 e^{3x}\,dx$$

Let $f(x) = e^{3x}\ \forall x \in [0,1]$ & $g(x) = \frac{1}{3}e^{3x}\ \forall x \in [0,1]$

$\Rightarrow f(x)$ is Continuous on $[0,1]$ & hence Integrable.

$\Rightarrow g(x)$ is derivable on $[0,1]$ & $g'(x) = f(x)$

$\therefore \int_0^1 e^{3x}\,dx = g(1) - g(0) = \frac{e}{3} - \frac{1}{3} = \frac{1}{3}(e-1)$.

Mean-Value theorem:-

If f is Continuous on $[a,b]$, then there exists $c \in (a,b)$ Such that $\int_a^b f(x)\,dx = f(c)(b-a)$

Proof: f is continuous on $[a,b] \Rightarrow f$ is bounded on $[a,b]$ & $f \in R[a,b]$

If m, M be the inf & Sup of f on $[a,b]$, we know that

$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$\therefore \exists\, \mu \in [m,M] \ni \int_a^b f(x)\,dx = \mu(b-a)$

Since, f is continuous on $[a,b]$ & $\mu \in [m,M] \ni$

$c \in (a,b) \ni f(c) = \mu$

$\therefore \int_a^b f(x)\,dx = f(c)(b-a)$.

now that $\frac{1}{4} < \int_0^{1/4} \frac{dx}{\sqrt{1-x^2}} < \frac{1}{\sqrt{15}}$

Sol: consider, $f(x) = \frac{1}{\sqrt{1-x^2}}$ on $[0, \frac{1}{4}]$

$(1-x^2)$ is Continuous on $\mathbb{R}$ and hence on $[0, \frac{1}{4}]$

$\sqrt{1-x^2}$ is Continuous on $[0,4]$, Since, $1-x^2 > 0$ on $[0, \frac{1}{4}]$

$1-x^2 \neq 0$ on $[0, 1/4] \Rightarrow f(x) = \frac{1}{\sqrt{1-x^2}}$ is continuous on $[0, 1/4]$

$\Rightarrow f(x)$ is integrable on $[0, 1/4]$.

By Mean-Value theorem, there exist $c \in (0, 1/4)$ Such that

$$\int_0^{1/4} \frac{dx}{\sqrt{1-x^2}} = f(c)\left[\frac{1}{4} - 0\right] = \frac{1}{4} \frac{1}{\sqrt{1-c^2}}$$

$$0 < c < \frac{1}{4} \Rightarrow 0 < c^2 < \frac{1}{16} \Rightarrow 0 > -c^2 > -\frac{1}{16}$$

$$\Rightarrow 1 > 1-c^2 > 1 - \frac{1}{16}$$

$$\Rightarrow 1 > \sqrt{1-c^2} > \frac{\sqrt{15}}{4}$$

$$\Rightarrow 1 < \frac{1}{\sqrt{1-c^2}} < \frac{4}{\sqrt{15}}$$

$$\Rightarrow \frac{1}{4} < \frac{1}{4\sqrt{1-c^2}} < \frac{1}{\sqrt{15}}$$

$$\Rightarrow \frac{1}{4} < \int_0^{1/4} \frac{dx}{\sqrt{1-x^2}} < \frac{1}{\sqrt{15}}.$$

First Mean Value theorem :-

If $f, g \in R[a,b]$ & g keeps the Same Sign on $[a,b]$, then $\exists\, \mu \in \mathbb{R}$ lying between the Infimum and Supremum of f Such that $\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx$.

Proof: Let g be non-negative on $[a,b]$.

then $g(x) \geq 0 \ \forall\, x \in [a,b]$

$f \in R[a,b] \Rightarrow f$ is bounded on $[a,b]$

$\Rightarrow m \leq f(x) \leq M \ \forall\, x \in [a,b]$

Let $m =$ Inf of f on $[a,b]$

$M =$ Sup of f on $[a,b]$

Since $g(x) \geq 0 \ \forall x \in [a,b]$

$\therefore m \leq f(x) \leq M \Rightarrow m\,g(x) \leq f(x)g(x) \leq M\,g(x)$

$\Rightarrow \int_a^b m\,g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \int_a^b M\,g(x)\,dx$

$\Rightarrow m\int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq M\int_a^b g(x)\,dx$

$\therefore \exists\, \mu \in [m,M] \ni \int_a^b f(x)\,g(x)\,dx = \mu\int_a^b g(x)\,dx$

Let $g(x)$ be non-~~negative~~ +ve ie; $g(x) \leq 0 \ \forall x \in [a,b]$

$\therefore m \leq f(x) \leq M \Rightarrow m\,g(x) \geq f(x)\cdot g(x) \geq M\,g(x)$

$\therefore m\int_a^b g(x)\,dx \geq \int_a^b g(x)\,f(x)\,dx \geq M\int_a^b g(x)\,dx \quad \forall x \in [a,b]$

$\Rightarrow \int_a^b m\,g(x)\,dx \geq \int_a^b M\,g(x)\,dx$

$\Rightarrow M\int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq m\int_a^b g(x)\,dx$

$\therefore \exists\, M \in [m,M] \ni \int_a^b f(x)g(x)\,dx = \mu\int_a^b g(x)\,dx$

** $\Rightarrow$ Prove that $\frac{1}{\pi} \leq \int_0^1 \frac{\sin\pi x}{1+x^2}\,dx \leq \frac{2}{\pi}$

Sol Take $f(x) = \frac{1}{1+x^2}$ and $g(x) = \sin\pi(x)$

Clearly, f, g are continuous on $[0,1]$ & hence integrable on $[0,1]$.

Also, $g(x) = \sin\pi x$ is positive on $(0,1)$

Since f is decreasing on $(0,1)$

Inf $f = f(1) = \frac{1}{2}$ & Sup $f = f(0) = 1$

$\therefore$ By first mean value theorem $\exists\, \mu \in [\frac{1}{2}, 1] \ni$

$\int_0^1 \frac{\sin\pi x}{1+x^2}\,dx = \mu\int_0^1 \sin\pi x\,dx = f(\xi)\int_0^1 \sin\pi x\,dx$

where $\xi \in (0,1)$

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fundamental Theorem, $\int_0^1 \sin \pi x \, dx = \frac{2}{\pi}$

$\therefore \int_0^1 \frac{\sin \pi x}{1+x^2} dx = f(\xi_1) \cdot \frac{2}{\pi}$, where $0 < \xi_1 < 1$

$0 \le \xi_1 \le 1 \Rightarrow \frac{1}{2} \le f(\xi_1) \le 1 \Rightarrow \frac{1}{2} \cdot \frac{2}{\pi} \le \frac{2}{\pi} f(\xi_1) \le 1 \cdot \frac{2}{\pi}$

$$\therefore \frac{1}{\pi} \le \int_0^1 \frac{\sin \pi x}{1+x^2} dx \le \frac{2}{\pi}.$$

* Using first mean-value theorem of Integral, prove that

$$x > \log(1+x) > \frac{x}{1+x} \; ; \; x \ge 0$$

Sol: let $f(x) = \frac{1}{1+x}$ and $g(x) = 1$ on $[0, t]$

Clearly,

f, g are bounded & Integrable on $[0, t]$ & g Keeps the same Sign in $[0, t]$

$0 < x < t \Rightarrow 1 < 1+x < 1+t \Rightarrow 1 > \frac{1}{1+x} > \frac{1}{1+t}$

By first mean value theorem,

$\int_0^t f(x) g(x) \, dx = \mu \int_0^t g(x) \, dx$, where μ is a number between the bounds of f.

$$\therefore \int_0^t \frac{1}{1+x} dx = \mu \int_0^t 1 \, dx = \mu t$$

$$\Rightarrow \log(1+t) = \mu t, \text{ where } \frac{1}{1+t} < \mu < 1$$

for $t \ge 0$, $\frac{1}{1+t} < \mu < 1 \Rightarrow \frac{1}{1+t} < \frac{\log(1+t)}{t} < 1$

$$\Rightarrow \frac{1}{1+t} < \log(1+t) < t.$$

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..gration By parts:-

Th: If f, g are derivable function on $[a,b]$ and f', g' are continuous on $[a,b]$ then $\int_a^b f(x) g'(x)\,dx$

$= f(b)g(b) - f(a)g(a) - \int_a^b f'(x) g'(x)\,dx$

Proof: f, g are derivable on $[a,b] \Rightarrow f, g$ are continuous on $[a,b] \Rightarrow f, g, fg \in R[a,b]$

f', g' are continuous on $[a,b] \Rightarrow f', g' \in R[a,b]$

$f \in R[a,b], g' \in R[a,b] \Rightarrow fg' \in R[a,b]$

$f' \in R[a,b], g \in R[a,b] \Rightarrow f'g \in R[a,b]$

$\therefore \int_a^b f(x) g'(x)\,dx$ & $\int_a^b f'(x) g'(x)\,dx$ exists

Now, $(fg)'(x) = f(x) g'(x) + f'(x) g(x) \in [a,b]$

$\therefore$ fg is a primitive of $fg' + gf'$ on $[a,b]$

$\therefore \int_a^b (f(x) g'(x) + f'(x) g(x))\,dx = (fg)b - fg(a)$

$\Rightarrow \int_a^b f(x) g'(x)\,dx = f(b) g(b) - f(a) g(a) - \int_a^b f'(x) g(x)\,dx$

* Prove that $\frac{\pi^3}{24} \leq \int_0^\pi \frac{x^2}{5+3\cos x}\,dx \leq \frac{\pi^3}{6}$.

Sol: let $g(x) = x^2$ & $f(x) = \frac{1}{5+3\cos x}$

g is continuous on $[0,\pi]$ & hence Integrable on $[0,\pi]$

$\Rightarrow f$ is Integrable on $[0,\pi]$

$g(x) \geq 0 \ \forall x \in [0,\pi] \Rightarrow g$ keeps same sign $[0,\pi]$

$\int_0^\pi g(x)\,dx = \int_0^\pi x^2\,dx = \left[\frac{x^3}{3}\right]_0^\pi = \frac{\pi^3}{3}$

$\therefore$ By first mean value theorem $\exists\, \mu \in [m, M]$ where

$m = \text{Inf } f$ on $[0,\pi]$ &

$M = \text{Sup } f$ on $[0,\pi]$

$$\int_0^{\pi} f(x)g(x)dx = \mu\int_0^{\pi} g(x)\,dx$$

$$\Rightarrow \int_0^{\pi} \frac{x^2}{5+3\cos x}\,dx = \mu\,\frac{\pi^3}{3} \quad \text{— (1)}$$

$\Rightarrow$ we have $-1 \le \cos x \le 1$

$$\Rightarrow -3 \le 3\cos x \le 3$$

$$\Rightarrow 5-3 \le 5+3\cos x \le 5+3$$

$$\Rightarrow 2 \le 5+3\cos x \le 8$$

$$\Rightarrow \frac{1}{2} \geq \frac{1}{5+3\cos x} \geq \frac{1}{8} \quad \forall\, x \in [0,\pi]$$

$\therefore m = \frac{1}{8}, \; M = \frac{1}{2}$

$\mu \in [m, M] \Rightarrow m \le \mu \le M$

$$\Rightarrow \frac{1}{8} \le \mu \le \frac{1}{2}$$

$$\Rightarrow \frac{\pi^3}{24} \le \mu\,\frac{\pi^3}{3} \le \frac{\pi^3}{6}$$

$$\Rightarrow \frac{\pi^3}{24} \le \int_0^{\pi} \frac{x^2}{5+3\cos x}\,dx \le \frac{\pi^3}{6} \quad (\text{from (1)})$$

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